Incorporation of Poisson Jumps into a model with nonparametrically estimated diffusion and application on the CEE countries' exchange rate.

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In the proposed model I extend the Ait-Sahalia (1996) model of nonparametric estimation of diffusion function by incorporating Poisson jumps. This extension should increase the precision of estimated diffusion function. Because of limited data available on Central European countries, the more complex Bandi and Nguyen (2000) approach cannot be used. The model will be calibrated on four Central European currencies. The CEE currency should benefit from the presence of Poison jump component because the emerging markets exhibit significantly lower liquidity than the advanced ones. In addition, the importance (and relevance) of a new component will be evaluated and discussed.

Key Words:

1. INTRODUCTION

The econometric models which use just Brownian motion as an error term are not able to explain certain stylized facts of financial time series, namely fat tails or high skewness. These models, therefore, do not mimic real data and consequently missprice derivative based on these models.

The most obvious way how to incorporate fat tails into an assets return distribution is to include a jump component. In addition, the excess kurtosis can arise either from volatile volatility or from a substantial jump component, which again motivates the use of a jump model.

One of the first models with Poisson jumps is Merton (1969). Jump process has been studied in the literature on term structure of interest rate and also in the foreign exchange literature, e.g. Ball and Roma (1994), Ball and Torous (1983), Jorion (1988) or Bates (1996).

In original Merton (1969) model the log of jump size is assumed to have a normal distribution with a constant mean value. However, the jump size distribution could be time varying if the conditional mean of the jump size distribution is allowed to be sensitive to trends in the market. Ball and Roma (1993) enriched the Vasicek (1977) model by time varying jump diffusion. For the purposes of their study, the jump size was modeled as the function of displacement from the central parity of EMS currencies. Bates and Craine (1999) specify the model where the volatility factor drives the intensity of jumps. Also Bates (2000) uses a time-varying arrival rate of jumps. All these mentioned authors assume the parametric specification of jump component.

One of the first papers dealing with nonparametric estimations of diffusion function of interest rate process is Ait-Sahalia (1996). He estimates the diffusion function nonparametrically while drift is still parametric. Stanton (1997) extends this paper and present methodology for estimation of both drift and diffusion nonparametrically. None of these two papers assume jump component. Bandi and Nguyen (2000) extend methodology event further and provide complete asymptotic theory for nonparametric estimates of drift, diffusion and jump intensity functions. Their paper is based on Johannes (2000) where he justifies the nonparametric extraction of the parameters and functions controlling the arrival of a jump from the estimated infinitesimal conditional moments.

However, there is one important problem connected with otherwise general methodology of Ait-Sahalia, Stanton or Johannes, namely data requirement. Ait-Sahalia (1996) uses 5500 observations (around 20 years), and Johannes (2000) or Bandi and Nguyen (2000) use more than 8000. For the purpose of the simulations they use 10000 observations, which is equal to 40 years of daily date. Since I want to estimate the model using exchange rate of Czech republic, I have just 2500 daily observations which covers approximately 10 years. The scarcity of data for this country is similar to any other transition country. This specific problem -relative small data sample- is a reason why I want to develop methodology which would produce estimates with lower standard error than known estimation techniques..

More specifically, my first goal is to develop estimation technique which would not be very data demanding and which would, at the same time, allow for nonparametric estimation of diffusion and jump component. In my paper I want to extend original model of Ait-Sahalia model (or simplify Johannes one) and asses the importance of this extension on advanced but primarily on emerging markets. It has been recognized in the finance literature that one of the most important features for derivative pricing is the specification of the diffusion function. Therefore, the inclusion of jump diffusion should improve the estimate of this function, and consequently increase the precision of derivative pricing. My methodology originates in papers which model short-term interest rates. In order to better illustrate my contribution and not to mix model for interest rate with model for exchange rate evolution, I will derive my methodology for interest rate.

The second goal is to put this methodology into exchange rate context. An important issue in this paper is to study whether seemingly higher volatility in emerging markets is captured by the models of price process with a jump component.

The rest of this paper is organized as follows. The Section 2 is devoted to the literature review. The specification of my model is in Section 3 and solution in Section 4. In Section 5 I describe the empirical results. Brief conclusion is at the end.

2. OVERVIEW OF LITERATURE

2.1. Models with parametric drift, diffusion functions and Poisson jumps.

In general, continuous time models in finance typically rest on one or more stationary diffusion processes with dynamics represented by Itô stochastic differential equation. The evolution of interest rate is governed by the process:

$$dr_t = \mu\left(r_t\right)dt + \sigma\left(r_t\right)dW_t$$

where functions $\mu(\cdot)$ and $\sigma(\cdot)$ are drift and diffusion functions respectively, $\{W_t, t \geq 0\}$ is a standard Brownian motion. Usually drift and diffusion is parameterized, $\mu(r,\theta)$ and $\sigma(r,\theta)$, $\theta \in \Theta \subset R^K$.

Using parametric specification has, definitely, its advantages. We can express process analytically and employ the maximum likelihood estimation procedure to obtain the estimates of parameters. However, it has also disadvantages – possible misspecification of the model. There is no reason to prefer one functional form of drift or diffusion to another.

Example of parametric models with Poisson jumps is Das's (1999) model, which comprises of mean-reverting drift, diffusion and Poisson Jump process. This model may be written

$$dr_t = \kappa\left(\theta - r_t\right)dt + \sigma dW_t + \xi(t)dJ\left(\lambda\right)$$

where ξ is a random jump whose size has a lognormal distribution with constant mean and volatility, and the arrival of jumps is governed by a Poisson process with arrival frequency parameter λ.This parameter indicates the the number of jumps per year. The diffusion and Poisson process are independent of each other, and independent of ξ as well. The return evolves with a mean-reverting drift and two random terms, one is Brownian diffusion and the other a Poisson process with random jumps ξ.

2.2. Model with nonparametric drift and diffusion (no jump)

The building stone of nonparametric estimation in finance is approach described in Ait-Sahalia (1996). He uses the parametric drift (instantaneous mean) and the nonparametric diffusion (or instantaneous variance) functions to model interest rate behavior. After estimating drift function parametrically and marginal density nonparametrically, the function of diffusion is constructed. Detail description of the model and estimation approach is in Appendix. Ait-Sahalia (1996) uses following model:

$$dr_t = \mu(r_t)\,dt + \sigma(r_t)\,dW_t$$

where $\{W_t, t \geq 0\}$ is a standard Brownian motion. $\mu(\cdot)$ and $\sigma(\cdot)$ are the drift and the diffusion functions of the process.

Let $\pi(\cdot)$ be the marginal density of the spot rate, and $\mu(\cdot)$ estimated parametric the drift. Diffusion function is then reconstructed by formula:

$$\sigma^2(r) = \frac{2}{\pi(r)} \int_o^r \mu(u,\theta)\,\pi(u)\,du$$

This equation shows that diffusion function can be constructed from the marginal distribution $\pi(\cdot)$, and the drift parameter vector θ. The identification of θ will be based on drift function. Ait-Sahalia (1996) assumes drift as a linear function of r_t.

Stanton (1997) develops a procedure for estimating *both functions* $-\mu(\cdot)$ and $\sigma^2(\cdot)$ – nonparametrically from data observed only at discrete time intervals. He uses Taylor expansions to construct a family of *approximations* to the drift, diffusion, and market price of risk functions. Stanton's estimate of drift confirms the hypothesis formulated in Lo and Wang (1995) about the misspecification of drift. He finds that the drift $\mu(\cdot)$, he estimated, shows evidence of substantial nonlinearity as opposed to linearly restricted as in Ait-Sahalia (1996). The estimated $\sigma(\cdot)$ is similar to the estimated (parametrically) by Chan, Karolyi, Longstaff and Sanders (1992).

In order to price assets, which depend on some underlying state variable, we need to know not only the dynamics of that variable, but also associated market price of risk (the excess return required by an investor to bear each extra unit of risk). Most of previous papers assume that this value is equal to 0, because of the local expectation hypothesis formulated in Cox, Ingersoll and Ross (1981). Stanton (1997) nonparametrically estimates the market price of risk as the function of level of interest rate. He finds out that this function is neither constant nor linear.

2.3. Model with nonparametric drift, diffusion and jump intensity

Bandi and Nguyen (2000) generalize the Johannes (2000) methodology for nonparametric estimation of drift, diffusion and jump intensity functions. Since the Bandi and Nguyen paper presents mainly the asymptotic theory, I will describe the Johannes methodology.

Consider a transformation of the process into logarithms

$$d\log(r_t) = \mu(r_{t-1})\,dt + \sigma(r_{t-1})\,dW_t + \xi dJ_t$$

The key to estimation is to identify the characteristics of the jump-diffusion dynamics through instantaneous moment conditions. Under regularity conditions:

1M $\quad \lim\limits_{\triangle\downarrow 0} \frac{1}{\triangle} E\left[\log\left(\frac{r_{t+\triangle}}{r_t}\right) \mid r_t = r\right] = \mu(r)$

2M $\quad \lim\limits_{\triangle\downarrow 0} \frac{1}{\triangle} E\left[\log\left(\frac{r_{t+\triangle}}{r_t}\right)^2 \mid r_t = r\right] = \sigma^2(r) + \lambda(r) E\left[\xi^2\right]$

4M $\quad \lim\limits_{\triangle\downarrow 0} \frac{1}{\triangle} E\left[\log\left(\frac{r_{t+\triangle}}{r_t}\right)^4 \mid r_t = r\right] = 5\lambda(r)\left(\sigma_\xi^2\right)^2$

6M $\quad \lim\limits_{\triangle\downarrow 0} \frac{1}{\triangle} E\left[\log\left(\frac{r_{t+\triangle}}{r_t}\right)^6 \mid r_t = r\right] = 15\lambda(r)\left(\sigma_\xi^2\right)^3$

The identification scheme uses the fact that the 1st, 2nd, 4th and 6th moments identify $\mu(r), \sigma^2(r), \lambda(r), \sigma_\xi^2$. To see this, the 4th and 6th moments completely identify the jump components. Given jump components, the second moment identifies the diffusion coefficient, $\sigma^2(r)$, and the first moment identifies the drift. For estimation of the conditional j-th moments, he uses nonparametric kernel estimators in a following form:

$$M^j = \frac{\frac{1}{h}\sum_{i=1}^{n-1} K\left(\frac{X_i - a}{h}\right)\left[X_{i+1} - X_i\right]^k}{\frac{1}{h}\sum_{i=1}^{n} K\left(\frac{X_i - a}{h}\right)}, \text{ if } \triangle = 1$$

Bandi and Nguyen (2000) prove that the estimation scheme outlined above is consistent.

3. MODEL

My model originates in Ait-Sahalia (1996) and Johannes (2000). I adopt parametric drift and also jump intensity in order to save the nonparametric estimation for diffusion function.

My model for interest rate evolution is following

$$d\log(r_t) = \mu(r_{t-1})\,dt + \sigma(r_{t-1})\,dW_t + \xi dJ_t$$

where $\{W_t, t \geq 0\}$ is a scalar standard Brownian motions, $\mu(r)$ is a drift function and $\sigma(r)$ is a diffusion function. The jumps arrive with intensity $\lambda(r)$ and the jump sizes are assumed to be normally distributed $\xi \sim N\left(0, \sigma_\xi^2\right)$. It is important assumption that mean jump size is 0. For some assets it could be problematic, since jumps usually move price in a certain direction. However, in exchange rate context is quite acceptable, because I assume that jumps just add volatility to the price process rather than change drift of the exchange rate. Moreover, specifying the process in logarithms with mean zero jumps ensures that $\mu(r)$ retains its interpretation as the local mean of the process.

4. SOLUTION

4.1. Jump intensity is constant

Let us assume $\lambda(r) = \lambda = const$. In the presence of jumps and constant jump intensity, the shape of the diffusion function will be the same as without jumps, but will be shifted downward. The exact formula for diffusion function is

$$\sigma^2(r) = \sigma_T^2(r) - \lambda\sigma_\xi^2,$$

where is $\sigma^2(r)$ the diffusion from continuous part, σ_ξ^2 is diffusion from discontinuous part and $\sigma_T^2(r)$ is the total diffusion.

The intuition behind this result is following. The jump size and jump intensity are not conditioned on the actual level of interest rate, r, and, therefore, they have no influence on the shape of the diffusion function. They have rather influence on the scale of this function. Since both diffusions have different impact on option prices, I should achieve higher precision of option pricing.

The total diffusion can be estimated as in Ait-Sahalia (1996), where $\sigma_T^2(r) = \dfrac{2}{\pi(r)} \displaystyle\int_o^r \mu(u,\theta)\,\pi(u)\,du$ The problem, however, arrises when we want to estimate the parameters of jumps, λ and σ_ξ^2. Estimation of these two parameters from the model assumed in Ait-Sahalia is complicated. Moreover, the log-transformation of the interest rate process used in Johannes prevents me from using his methodology for estimation of λ and σ_ξ^2. In this case, estimates from log-model would be different from estimates from original model.

Therefore, my proposed methodology originates in Johannes (2000):

1. Estimate parametrically the drift $\mu(\cdot)$. Since I assume linear mean-reverting function of drift, ordinary least square identifies the parameters α and β in $E[\log r_{t+1} - \log r_t \mid r_t] = \alpha + \beta r_t$.

2. Estimate the λ and σ_ξ^2. Based of Johannes (2000) calculation of moments, ratio of 6th and 4th moment will give me the desired estimate of the σ_ξ^2. Consequently, the estimate of the λ will be $\lambda = \frac{4th\,Moment}{5\left(\sigma_\xi^2\right)^2}$. Particular moments will be estimated as follows: 4th Moment $= \frac{1}{n}\sum_{i=1}^{n-1}\log^4\left(\frac{r_{t+\triangle}}{r_t}\right) = 5\lambda\left(\sigma_\xi^2\right)^2$, and 6th Moment $= \frac{1}{n}\sum_{i=1}^{n-1}\log^6\left(\frac{r_{t+\triangle}}{r_t}\right) = 15\lambda\left(\sigma_\xi^2\right)^3$.

3. Diffusion function can be completely identified from the 2nd moment estimated nonparametrically, e.g. 2nd Moment $= \lim_{\triangle\downarrow 0} \frac{1}{\triangle} E\left[\log\left(\frac{r_{t+\triangle}}{r_t}\right)^2 \mid r_t = r\right]$. Therefore, the $\sigma^2(r) = 2nd\ Moment\,(r) - \lambda\sigma_\xi^2$.

4. The diffusion estimator can be used to correct for heteroskedasticity in the residuals from the regression in step 1.

4.2. Jump intensity is not constant

Now, I will assume that the jump intensity is not constant but rather depend (in a certain way) on r. Naturally, I will assume the parametric specification. Using parametric specification will allow me to use the smaller data sample and, at the same time, benefit from good properties of models which have nonparametric diffusion function and jump component.

A first candidate for parametric specification of $\lambda(\cdot)$ is linear function, namely $\lambda(r) = a + br$. This specification looks simple, but if we would take to account the error band, the Johannes nonparametric estimates are not critically far from linear function. By letting $b = 0$, the linear specification collapses to the previous case of constant jump intensity. In this case $\sigma^2(r)$ will have a different shape than in case without jumps.

The estimation of all function of interest rate process is following

1. Estimate parametrically the drift $\mu(\cdot)$. I can assume linear mean-reverting function of drift, ordinary least square identifies the parameters α and β in $E[\log r_{t+1} - \log r_t \mid r_t] = \alpha + \beta r_t$.

2. Estimate the $\lambda(r)$ and σ_ξ^2. Based of Johannes (2000) calculation of moments, the ratio of 6th and 4th moment will give me the desired estimate of the σ_ξ^2. Consequently, the parameters of $\lambda(r) = a + br$ will be identified via regression $\frac{E\left[\log^4\left(\frac{r_{t+\triangle}}{r_t}\right) \mid r_t = r\right]}{5\left(\sigma_\xi^2\right)^2} = a + br_t.$

3. Diffusion function can be completely identified from the 2nd moment estimated nonparametrically, e.g. 2nd Moment $= \lim_{\triangle\downarrow 0} \frac{1}{\triangle} E\left[\log\left(\frac{r_{t+\triangle}}{r_t}\right)^2 \mid r_t = r\right]$. Therefore, the $\sigma^2(r) = 2nd\ Moment\,(r) - \lambda(r)\,\sigma_\xi^2$.

4. The diffusion estimator can be used to correct for heteroskedasticity in the residuals from the regression in step 1.

4.3. Simultaneous estimation of Jump intensity and diffusion

In Appendix, I present the nonparametric estimates of drift, diffusion and jump intensity functions from Johannes (2000). I find that the shape $\sigma^2(r)$ is similar to shape of $\lambda(r)$. Therefore, I propose to use the functional form of diffusion function for estimating the jump intensity function, e.g. $\lambda(r) = f(\sigma(r))$ or $\lambda(r) = \sigma(f(r))$ The function f will have a simple linear form, $f(x) = a + bx$, which will imply that $\lambda(r) = a + b\sigma^2(r)$. If the parameter b would be insignificant, jump intensity would be $\lambda(r) = a = const$. The later transformation, $\lambda(r) = \sigma(f(r))$, however, is not suitable for the following reason: $\lambda(r)$ could have a different scale than $\sigma^2(r)$. Therefor I will adopt the former version of transformation, namely $\lambda(r) = f(\sigma^2(r))$.

Naturally, estimation of $\lambda(r)$ and $\sigma^2(r)$ will have to be done simultaneously. This kind of estimation will restrict (or modify) the shape of $\sigma^2(r)$, but I will gain on avoiding misspecification of Poisson intensity function. This method could preserve high complexity of functions and save data requirements at the same time.

Derivation of the estimates starts with Johannes instantaneous moments conditions:

$$2M(r) = \sigma^2(r) + \lambda(r)\sigma_\xi^2$$
$$4M(r) = 5\lambda(r)\left(\sigma_\xi^2\right)^2$$

Let's denote the shape $\sigma^2(r)$ as $f(r)$. Provided that $\lambda(r) = a + b\sigma^2(r)$ and estimate of σ_ξ^2 is known,

$$2M(r) = f(r) + (a + bf(r))\sigma_\xi^2$$
$$4M(r) = 5(a + bf(r))\left(\sigma_\xi^2\right)^2 \qquad / : 5\sigma_\xi^2$$

Rearranging the terms

$$2M(r) = f(r)\left(1 + \sigma_\xi^2 b\right) + a\sigma_\xi^2$$
$$\frac{4M(r)}{5\sigma_\xi^2} = (a + bf(r))\sigma_\xi^2 = f(r)\left(\sigma_\xi^2 b\right) + a\sigma_\xi^2$$

By subtracting of the first equation from the second one, I will get the estimate of $f(r) = 2M(r) - \frac{4M(r)}{5\sigma_\xi^2}$.

To write it precisely,

$$\lim_{\triangle\downarrow 0}\frac{1}{\triangle}E\left[\log\left(\frac{r_{t+\triangle}}{r_t}\right)^2 - \frac{\log\left(\frac{r_{t+\triangle}}{r_t}\right)^4}{5\sigma_\xi^2} \mid r_t = r\right] = f(r)$$

From the equation for second instantaneous moments moment, I will get estimates of parameters a and b and hence the $\lambda(r)$.

5. EMPIRICAL ANALYSIS

5.1. Comparative studies

In the first part of empirical section, I will calibrate my new model with Ait-Sahalia data (US interest rate). I will compare the diffusion function from my model with those presented in Ait-Sahalia. The differences in shape of these two functions will enable me to assess the effect of Poisson jumps. Direct comparison of my results with results presented in paper, however, will not be possible, since in my methodology I assume log-transformation of Ait-Sahalia process. Anyway, I will evaluate the importance of generalization. On the other hand, calibration of model on Johannes (2000) data will assess the loss coming from parametric restriction imposed on drift and jump intensity.

5.2. Exchange rate

The prime interest of my paper is modelling of the exchange rates of the Central European countries. I will calibrate the model with the Czech koruna exchange rates. The effect of Poisson process should be more pronounced than in the advanced markets.

Since I will work in exchange rate context, I need to model drift as a function of instantaneous expected rate of appreciation of the foreign currency which is equal to the interest rate differential.

My model for exchange rate has following specification:

$$d\log S = \mu\left(b\right)dt + \sigma\left(b\right)dW_t + \xi dJ$$
$$prob\left(dJ = 1\right) = \lambda\left(b\right)dt \text{ and } \xi \sim N\left(0, \sigma_\xi^2\right)$$

where $\{W_t, t \geq 0\}$ is a standard Brownian motion, S is a nominal price of foreign currency in terms of domestic one; $b = r^{domestic} - r^{foreign}$ is interest rate differential; $\mu\left(b\right)$ is parametric mean reverting drift function; and $\sigma\left(b\right)$ is nonparametric diffusion function.

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